

AN INDEFINITE LAPLACIAN ON A RECTANGLE

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ABSTRACT. In this note we investigate the nonelliptic differential expression $\mathcal{A} = -\operatorname{div} \operatorname{sgn} \nabla$ on a rectangular domain Ω in the plane. The seemingly simple problem to associate a selfadjoint operator with the differential expression \mathcal{A} in $L^2(\Omega)$ is solved here. Such indefinite Laplacians arise in mathematical models of metamaterials characterized by negative electric permittivity and/or negative magnetic permeability.

1. INTRODUCTION

Consider the domains $\Omega_+ = (0, 1) \times (0, 1)$ and $\Omega_- = (-1, 0) \times (0, 1)$ and let $\Omega = (-1, 1) \times (0, 1)$ and $\mathcal{C} = \{0\} \times (0, 1)$. We study the nonelliptic differential expression \mathcal{A} defined by

$$(1.1) \quad \mathcal{A}f = -\operatorname{div}(\operatorname{sgn} \nabla f), \quad \text{where} \quad \operatorname{sgn}(x, y) = \begin{cases} 1, & (x, y) \in \Omega_+, \\ -1, & (x, y) \in \Omega_-, \end{cases}$$

on the rectangle Ω . Our aim is to associate a selfadjoint operator in $L^2(\Omega)$ with Dirichlet boundary conditions on $\partial\Omega$ to \mathcal{A} . Informally speaking, in this seemingly simple toy problem this will be the partial differential operator

$$(1.2) \quad \begin{aligned} Af &= \mathcal{A}f = \begin{pmatrix} -\Delta f_+ \\ \Delta f_- \end{pmatrix}, \\ \operatorname{dom} A &= \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : \begin{array}{l} f_{\pm}, \Delta f_{\pm} \in L^2(\Omega_{\pm}), f|_{\partial\Omega} = 0, \\ f_+|_{\mathcal{C}} = f_-|_{\mathcal{C}}, \partial_{\mathbf{n}_+} f_+|_{\mathcal{C}} = \partial_{\mathbf{n}_-} f_-|_{\mathcal{C}} \end{array} \right\}, \end{aligned}$$

where f_{\pm} denote the restrictions of a function $f \in L^2(\Omega)$ onto Ω_{\pm} , and the normal derivatives $\partial_{\mathbf{n}_+}$ and $\partial_{\mathbf{n}_-}$ point outward of Ω_{\pm} (and hence in opposite directions at \mathcal{C}). The main peculiarity here is the interface condition

$$\partial_{\mathbf{n}_+} f_+|_{\mathcal{C}} = \partial_{\mathbf{n}_-} f_-|_{\mathcal{C}}, \quad f = (f_+, f_-)^{\top} \in \operatorname{dom} A,$$

for the normal derivatives, which is due to the sign change and discontinuity of the coefficient sgn at \mathcal{C} . Our main result states that (when the Dirichlet and Neumann traces are properly interpreted) the operator A in (1.2) is selfadjoint in $L^2(\Omega)$.

The non-standard interface condition is responsible for unexpected spectral properties of A . Although the domain Ω is bounded, it turns out that the essential spectrum of A is not empty, namely 0 is an isolated eigenvalue of infinite multiplicity. The remaining part of the spectrum of A consists of discrete eigenvalues which accumulate to $+\infty$ and $-\infty$. We note that the differential equation $\mathcal{A}f = \lambda f$ can of course be solved by separation of variables; the main feature of this note is the description of the domain of the corresponding selfadjoint operator A with explicit boundary and interface conditions.

We point out that $\text{dom } A$ contains functions which do not belong to any local Sobolev space H^s , $s > 0$, in a neighbourhood of the interface \mathcal{C} . This leads to the following difficulties: Green's identity is not valid for functions $f, g \in \text{dom } A$ and the definition of the (local) Dirichlet and Neumann traces is rather subtle, and requires a particularly careful analysis. Here we employ recent results on the extension of trace maps onto maximal domains of Laplacians on (quasi-)convex and Lipschitz domains from [2, 12] and we rely on the description of the traces of $H^2(\Omega_\pm)$ -functions in [13]. It finally turns out that the operator A can be viewed as a kind of Krein-von Neumann extension of a non-semibounded symmetric operator with infinite defect and domain contained in $H^2(\Omega_+) \times H^2(\Omega_-)$; thus only the functions in the infinite dimensional eigenspace $\ker A$ do not possess H^s -regularity near the interface \mathcal{C} .

We wish to emphasize that our result complements the results in [4] where the related problem

$$(1.3) \quad \mathcal{A}_\varepsilon f = -\text{div}(\varepsilon \nabla f), \quad \varepsilon(x, y) = \begin{cases} \varepsilon_+, & (x, y) \in \Omega_+, \\ -\varepsilon_-, & (x, y) \in \Omega_-, \end{cases}$$

with $\varepsilon_\pm > 0$ was treated under the assumption $\varepsilon_+ \neq \varepsilon_-$ with the help of boundary integral methods on more general domains $\Omega \subset \mathbb{R}^2$; for related problems see also [3, 8, 9, 14, 19, 20]. It is shown in [4] that, if $\varepsilon_+ \neq \varepsilon_-$, the operator

$$(1.4) \quad A_\varepsilon f = \mathcal{A}_\varepsilon f, \quad \text{dom } A_\varepsilon = \{f \in H_0^1(\Omega) : \mathcal{A}_\varepsilon f \in L^2(\Omega)\},$$

is selfadjoint, has a compact resolvent, and with eigenvalues accumulating to $+\infty$ and $-\infty$. The borderline case $\varepsilon_+ = \varepsilon_-$ that we investigate in this note was excluded in [4] and the other works (except for the one-dimensional situation [20], which is intrinsically different). We also wish to mention that abstract representation theorems for indefinite quadratic forms and related form methods in [18] (see also [11, 19] and [25, 27]) are not directly applicable in the present problem or do not lead to a selfadjoint operator in $L^2(\Omega)$. The eigenvalue problem $\mathcal{A}_\varepsilon f = \lambda f$ in our rectangular geometry was previously considered in [19] with the help of separation of variables (cf. Section 5), from which it follows that 0 is an eigenvalue of infinite multiplicity provided that $\varepsilon_+ = \varepsilon_-$.

The indefinite differential expressions (1.1) and (1.3) arise in mathematical models of metamaterials which are characterized by negative electric permittivity and/or negative magnetic permeability (see [24, 26] for a physical survey and [5, 7, 10] for a rigorous justification of the models via a homogenization of Maxwell's equations in geometrically non-trivial periodic structures). More specifically, our rectangular model can be thought as simulating an interface between a dielectric material in Ω_+ and a metamaterial in Ω_- . It has been known since the seminal work [8] that the problem of the type $\mathcal{A}_\varepsilon f = \rho$ in Ω with a smooth interface is well posed in $H_0^1(\Omega)$ if and only if the contrast $\kappa := \varepsilon_+/\varepsilon_-$ is different from 1. Proving that (1.2) is selfadjoint, in this note we provide a correct functional setting for the problem on a rectangle in the critical situation $\kappa = 1$. Moreover, in Section 5 of this note we show that the eigenvalues and eigenfunctions of A_ε converge to eigenvalues and eigenfunctions of the operator A as $\kappa \rightarrow 1$.

An alternative approach to theoretical studies of metamaterials is to add a small imaginary number to the negative value of sgn , arguing that “real systems are

always slightly lossy”, see, e.g. [24]. This leads to a complexified differential expression

$$(1.5) \quad \mathcal{B}_\eta f = -\operatorname{div}(\varepsilon_\eta \nabla f), \quad \varepsilon_\eta(x, y) = \begin{cases} 1, & (x, y) \in \Omega_+, \\ -1 + i\eta, & (x, y) \in \Omega_-, \end{cases}$$

with $\eta > 0$, which immediately provides a well-defined operator

$$(1.6) \quad B_\eta f = \mathcal{B}_\eta f, \quad \operatorname{dom} B_\eta = \{f \in H_0^1(\Omega) : \mathcal{B}_\eta f \in L^2(\Omega)\}.$$

Indeed, the rotated operator $e^{-i(\pi/2-\eta)} B_\eta$ is an m -sectorial operator with vertex 0 and semi-angle $\pi/2 - \eta$, which is defined via the associated sectorial form defined on $H_0^1(\Omega)$; cf. [21, Sec. VI]. It follows that B_η is an operator with compact resolvent for every $\eta > 0$, albeit non-selfadjoint now. Let us note that considering the complexified problem $B_\eta f = \rho$ in the limit as $\eta \rightarrow 0$ is a conventional way how to describe the cloaking effects in metamaterials (of different geometric structure) through the “anomalous localized resonance”, see [6, 23]. We shall show that the eigenvalues and eigenfunctions of B_η converge to eigenvalues and eigenfunctions of our operator A as $\eta \rightarrow 0$. Recall that $Af = \rho$ is generally ill-posed since 0 is an eigenvalue of infinite multiplicity.

This note is organized as follows. In Section 2 we establish a modified version of Green’s identity and other preliminary results that we shall frequently use later. In Section 3 we introduce an auxiliary closed symmetric operator R and study its properties. By considering a generalized Krein–von Neumann extension of R , the selfadjointness of A is proved in Section 4, where we also discuss qualitative spectral properties of A . More quantitative results about the spectrum of A and the aforementioned convergence results are established in Section 5.

Acknowledgement. We wish to thank our colleagues Guy Bouchitté, E. Brian Davies, Amru Hussein, Vadim Kostrykin, Rainer Picard, Karl-Michael Schmidt, and Sascha Trostorff for fruitful discussions. This work is supported by the Austrian Science Fund (FWF), project P 25162-N26, Czech project RVO61389005 and the GACR grant No. 14-06818S.

2. A GENERALIZED GREEN’S IDENTITY ON THE MAXIMAL DOMAIN

The Dirichlet realizations $A_{D\pm}$ associated to $\mp\Delta$ in $L^2(\Omega_\pm)$ will play an important role in the sequel. Recall that

$$(2.1) \quad A_{D\pm} = \mp\Delta, \quad \operatorname{dom} A_{D\pm} = H_0^1(\Omega_\pm) \cap H^2(\Omega_\pm),$$

are selfadjoint operators in $L^2(\Omega_\pm)$ with compact resolvents, that A_{D+} is uniformly positive, and that A_{D-} is uniformly negative. Here the H^2 -regularity is consequence of Ω_\pm being convex; cf. [15, 16]. If γ_D denotes the Dirichlet trace operator defined on $H^2(\Omega_\pm)$ then one has

$$\operatorname{dom} A_{D\pm} = \{f_\pm \in H^2(\Omega_\pm) : \gamma_D f_\pm = 0\}.$$

The selfadjoint Neumann operators are given by

$$A_{N\pm} = \mp\Delta, \quad \operatorname{dom} A_{N\pm} = \{f_\pm \in H^2(\Omega_\pm) : \gamma_{N\pm} f_\pm = 0\},$$

where $\gamma_{N\pm}$ are the Neumann trace operator defined on $H^2(\Omega_\pm)$ with normal pointing outwards Ω_\pm .

We shall also make use of the spaces

$$\begin{aligned}\mathcal{G}_N(\partial\Omega_\pm) &:= \text{ran}(\gamma_{N_\pm}(\text{dom } A_{D\pm})) = \{\gamma_{N_\pm} f_\pm : f_\pm \in H^2(\Omega_\pm), \gamma_D f_\pm = 0\}, \\ \mathcal{G}_D(\partial\Omega_\pm) &:= \text{ran}(\gamma_D(\text{dom } A_{N\pm})) = \{\gamma_D f_\pm : f_\pm \in H^2(\Omega_\pm), \gamma_{N_\pm} f_\pm = 0\},\end{aligned}$$

which were characterized and denoted by $N^{1/2}(\partial\Omega_\pm)$ and $N^{3/2}(\partial\Omega_\pm)$, respectively, in [12], and also appear in [2] in a more general setting. We equip $\mathcal{G}_N(\partial\Omega_\pm)$ and $\mathcal{G}_D(\partial\Omega_\pm)$ with the natural norms [12, (6.6) and (6.42)]. If \mathbf{n}_\pm and \mathbf{t}_\pm denote the unit normal pointing outwards and a corresponding tangential vector, respectively, and $\partial_{\mathbf{t}_\pm}$ is the tangential derivative on $\partial\Omega_\pm$, then according to [13, Theorem 3] one has

$$(\gamma_{N_\pm} f_\pm) \mathbf{t}_\pm \in (H^{1/2}(\partial\Omega_\pm))^2$$

for all $\gamma_{N_\pm} f_\pm \in \mathcal{G}_N(\partial\Omega_\pm)$ and

$$(\partial_{\mathbf{t}_\pm} \gamma_D f_\pm) \mathbf{n}_\pm \in (H^{1/2}(\partial\Omega_\pm))^2$$

for all $\gamma_D f_\pm \in \mathcal{G}_D(\partial\Omega_\pm)$, where

$$H^{1/2}(\partial\Omega_\pm) = \left\{ \varphi \in L^2(\partial\Omega_\pm) : \int_{\partial\Omega_\pm} \int_{\partial\Omega_\pm} \frac{|\varphi(\alpha) - \varphi(\beta)|^2}{|\alpha - \beta|^2} d\alpha d\beta < \infty \right\}.$$

The following statement on the decomposition of functions in $\mathcal{G}_N(\partial\Omega_\pm)$ and $\mathcal{G}_D(\partial\Omega_\pm)$ in two parts with supports on \mathcal{C} and $\mathcal{C}_\pm := \partial\Omega_\pm \setminus \mathcal{C}$, respectively, is a direct consequence of the abovementioned fact.

Lemma 2.1. *Every function $\varphi \in \mathcal{G}_N(\partial\Omega_\pm)$ (resp. $\varphi \in \mathcal{G}_D(\partial\Omega_\pm)$) admits a decomposition in the form*

$$(2.2) \quad \varphi = (\varphi|_{\mathcal{C}})^\sim + (\varphi|_{\mathcal{C}_\pm})^\sim$$

where $(\varphi|_{\mathcal{C}})^\sim \in \mathcal{G}_N(\partial\Omega_\pm)$ (resp. $(\varphi|_{\mathcal{C}})^\sim \in \mathcal{G}_D(\partial\Omega_\pm)$) is the extension of $\varphi|_{\mathcal{C}}$ to $\partial\Omega_\pm$ by 0, and $(\varphi|_{\mathcal{C}_\pm})^\sim \in \mathcal{G}_N(\partial\Omega_\pm)$ (resp. $(\varphi|_{\mathcal{C}_\pm})^\sim \in \mathcal{G}_D(\partial\Omega_\pm)$) is the extension of $\varphi|_{\mathcal{C}_\pm}$ to $\partial\Omega_\pm$ by 0.

Consider the symmetric operators $S_\pm = \mp\Delta$, $\text{dom } S_\pm = H_0^2(\Omega_\pm)$, and their adjoints

$$(2.3) \quad S_\pm^* = \mp\Delta, \quad \text{dom } S_\pm^* = \{f_\pm \in L^2(\Omega_\pm) : \Delta f_\pm \in L^2(\Omega_\pm)\}.$$

Since $0 \notin \sigma(A_{D,\pm})$ one has the direct sum decompositions

$$(2.4) \quad \text{dom } S_\pm^* = \text{dom } A_{D\pm} \dot{+} \ker S_\pm^*.$$

In the following we will often decompose functions $f_\pm \in \text{dom } S_\pm^*$ accordingly, that is, we write

$$(2.5) \quad f_\pm = f_{D\pm} + f_{0\pm}, \quad f_{D\pm} \in \text{dom } A_{D\pm}, \quad f_{0\pm} \in \ker S_\pm^*.$$

It is also important to note that the spaces $\ker S_\pm^* \cap H^2(\Omega_\pm)$ are dense in $\ker S_\pm^*$, where the latter spaces are equipped with the L^2 -norm (or, equivalently with the graph norm of S_\pm^*). This fact can be shown with the help of the density result [12, (6.30)] for $s = 0$.

Recall from [12, Theorem 6.4] that the Dirichlet traces γ_D admit continuous and surjective extensions

$$\tilde{\gamma}_D : \text{dom } S_\pm^* \rightarrow (\mathcal{G}_N(\partial\Omega_\pm))^*,$$

where $\text{dom } S_{\pm}^*$ is equipped with the graph norm and $(\mathcal{G}_N(\partial\Omega_{\pm}))^*$ is the conjugate dual space of $\mathcal{G}_N(\partial\Omega_{\pm})$ equipped with the corresponding norm. It is important to note that

$$(2.6) \quad \ker \tilde{\gamma}_D = \ker \gamma_D = \text{dom } A_{D\pm} = H_0^1(\Omega_{\pm}) \cap H^2(\Omega_{\pm}),$$

where the first equality has been shown in [2, Section 4.1] and the other identities are clear from the above.

We shall denote the duality pairing between $\mathcal{G}_N(\partial\Omega_{\pm})$ and $(\mathcal{G}_N(\partial\Omega_{\pm}))^*$ in the form

$$\mathcal{G}_N(\partial\Omega_{\pm})^* \langle \psi, \varphi \rangle_{\mathcal{G}_N(\partial\Omega_{\pm})}, \quad \psi \in \mathcal{G}_N(\partial\Omega_{\pm})^*, \quad \varphi \in \mathcal{G}_N(\partial\Omega_{\pm}),$$

and occasionally we also write $\psi(\varphi)$ in this situation.

It will also be used later that the Neumann traces $\gamma_{N\pm}$ admit continuous and surjective extensions

$$\tilde{\gamma}_{N\pm} : \text{dom } S_{\pm}^* \rightarrow (\mathcal{G}_D(\partial\Omega_{\pm}))^* ;$$

this fact was observed in [12, Theorem 6.10]. Here again $\text{dom } S_{\pm}^*$ is equipped with the graph norm and $(\mathcal{G}_D(\partial\Omega_{\pm}))^*$ is the conjugate dual space of $\mathcal{G}_D(\partial\Omega_{\pm})$ equipped with the corresponding norm.

The next proposition shows that a modified Green's identity (with the Neumann trace $\gamma_{N\pm} f_{\pm}$ replaced by the regularized Neumann trace $\gamma_{N\pm} f_{D\pm}$) remains valid on the maximal domains $\text{dom } S_{\pm}^*$. This fact is essentially a consequence of [12, Theorem 6.4]. We also mention that analogous extensions of Green's identity are well known for elliptic operators on smooth domains, see, e.g. [17].

Proposition 2.2. *The following Green's identity holds for all $f_{\pm} = f_{D\pm} + f_{0\pm}$ and $g_{\pm} = g_{D\pm} + g_{0\pm}$ in $\text{dom } S_{\pm}^*$:*

$$\begin{aligned} & (S_{\pm}^* f_{\pm}, g_{\pm})_{L^2(\Omega_{\pm})} - (f_{\pm}, S_{\pm}^* g_{\pm})_{L^2(\Omega_{\pm})} \\ &= \pm \mathcal{G}_N(\partial\Omega_{\pm})^* \langle \tilde{\gamma}_D f_{\pm}, \gamma_{N\pm} g_{D\pm} \rangle_{\mathcal{G}_N(\partial\Omega_{\pm})} \mp \mathcal{G}_N(\partial\Omega_{\pm}) \langle \gamma_{N\pm} f_{D\pm}, \tilde{\gamma}_D g_{\pm} \rangle_{\mathcal{G}_N(\partial\Omega_{\pm})}. \end{aligned}$$

Proof. The identity will only be shown in $L^2(\Omega_+)$. The same argument applies on Ω_- . Let $f_+ = f_{D+} + f_{0+}$, $g_+ = g_{D+} + g_{0+} \in \text{dom } S_+^*$ and recall from [12, Theorem 6.4] that the identity

$$(S_+^* f_+, g_{D+}) - (f_+, A_{D+} g_{D+}) = \mathcal{G}_N(\partial\Omega_+)^* \langle \tilde{\gamma}_D f_+, \gamma_{N+} g_{D+} \rangle_{\mathcal{G}_N(\partial\Omega_+)}$$

holds, where we simply write (\cdot, \cdot) for the inner product in $L^2(\Omega_+)$. Since A_{D+} is selfadjoint in $L^2(\Omega_+)$ it is clear that

$$(A_{D+} f_{D+}, g_{D+}) - (f_{D+}, A_{D+} g_{D+}) = 0.$$

Moreover, as $f_{0+}, g_{0+} \in \ker S_+^*$ we also have

$$(S_+^* f_{0+}, g_{0+}) - (f_{0+}, S_+^* g_{0+}) = 0.$$

Taking this into account we compute

$$\begin{aligned} & (S_+^* f_+, g_+) - (f_+, S_+^* g_+) \\ &= (S_+^* (f_{D+} + f_{0+}), g_{D+} + g_{0+}) - (f_{D+} + f_{0+}, S_+^* (g_{D+} + g_{0+})) \\ &= (A_{D+} f_{D+}, g_{0+}) + (S_+^* f_{0+}, g_{D+}) - (f_{0+}, A_{D+} g_{D+}) - (f_{D+}, S_+^* g_{0+}) \\ &= (A_{D+} f_{D+}, g_{0+}) - (f_{D+}, S_+^* g_{0+}) + (S_+^* f_{0+}, g_{D+}) - (f_{0+}, A_{D+} g_{D+}) \\ &= -\mathcal{G}_N(\partial\Omega_+)^* \langle \gamma_{N+} f_{D+}, \tilde{\gamma}_D g_{0+} \rangle_{\mathcal{G}_N(\partial\Omega_+)^*} + \mathcal{G}_N(\partial\Omega_+)^* \langle \tilde{\gamma}_D f_{0+}, \gamma_{N+} g_{D+} \rangle_{\mathcal{G}_N(\partial\Omega_+)} \\ &= \mathcal{G}_N(\partial\Omega_+)^* \langle \tilde{\gamma}_D f_+, \gamma_{N+} g_{D+} \rangle_{\mathcal{G}_N(\partial\Omega_+)} - \mathcal{G}_N(\partial\Omega_+)^* \langle \gamma_{N+} f_{D+}, \tilde{\gamma}_D g_+ \rangle_{\mathcal{G}_N(\partial\Omega_+)^*}, \end{aligned}$$

where we have used $\ker \tilde{\gamma}_D = \ker \gamma_D$ from (2.6) in the last identity. \square

Next we consider the subspaces

$$\mathcal{G}_\pm := \{\varphi \in \mathcal{G}_N(\partial\Omega_\pm) : \varphi|_{\mathcal{C}} = 0\}$$

of $\mathcal{G}_N(\partial\Omega_\pm)$ which consist of functions vanishing on \mathcal{C} . Denote by $\mathcal{G}_\pm^\perp \subset (\mathcal{G}_N(\partial\Omega_\pm))^*$ the corresponding annihilators,

$$\mathcal{G}_\pm^\perp = \{\psi \in (\mathcal{G}_N(\partial\Omega_\pm))^* : \psi(\varphi) = 0 \text{ for all } \varphi \in \mathcal{G}_\pm\}.$$

Roughly speaking \mathcal{G}_\pm^\perp can be viewed as the linear subspaces of functionals from $(\mathcal{G}_N(\partial\Omega_\pm))^*$ that vanish on $\mathcal{C}_\pm = \partial\Omega_\pm \setminus \mathcal{C}$. It is important to note that

$$(2.7) \quad \mathcal{G}_\pm^\perp \cong (\mathcal{G}_N(\partial\Omega_\pm)/\mathcal{G}_\pm)^*.$$

In particular, if for some $\varphi \in \mathcal{G}_N(\partial\Omega_\pm)$ and all $\psi \in \mathcal{G}_\pm^\perp$ one has $\psi(\varphi) = 0$ then $\varphi = 0$ when identified with elements in the quotient space $\mathcal{G}_N(\partial\Omega_\pm)/\mathcal{G}_\pm$ and hence $\varphi \in \mathcal{G}_\pm$, that is, $\varphi|_{\mathcal{C}} = 0$.

3. AN AUXILIARY SYMMETRIC OPERATOR R

In the next proposition we consider a restriction R of the selfadjoint operator $A_{D+} \oplus A_{D-}$ in $L^2(\Omega)$ and we determine the adjoint of R . It will later turn out that the operator A in (1.2) is a selfadjoint extension of R (and hence a restriction of the adjoint operator R^*).

Proposition 3.1. *The operator*

$$Rf = \mathcal{A}f = \begin{pmatrix} -\Delta f_+ \\ \Delta f_- \end{pmatrix},$$

$$\text{dom } R = \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : f_\pm \in H^2(\Omega_\pm) \cap H_0^1(\Omega_\pm), \gamma_{N+} f_+|_{\mathcal{C}} = \gamma_{N-} f_-|_{\mathcal{C}} \right\},$$

is a closed symmetric operator with equal infinite deficiency indices in $L^2(\Omega)$ and $R \subset A_{D+} \oplus A_{D-}$ holds. The adjoint operator is given by

$$R^*f = \mathcal{A}f = \begin{pmatrix} -\Delta f_+ \\ \Delta f_- \end{pmatrix},$$

$$\text{dom } R^* = \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : f_\pm, \Delta f_\pm \in L^2(\Omega_\pm), \tilde{\gamma}_D f_\pm \in \mathcal{G}_\pm^\perp, \tilde{\gamma}_D f_+|_{\mathcal{C}} = \tilde{\gamma}_D f_-|_{\mathcal{C}} \right\},$$

where the boundary condition $\tilde{\gamma}_D f_+|_{\mathcal{C}} = \tilde{\gamma}_D f_-|_{\mathcal{C}}$ is understood as

$$\mathcal{G}_N(\partial\Omega_+)^* \langle \tilde{\gamma}_D f_+, \varphi \rangle_{\mathcal{G}_N(\partial\Omega_+)} = \mathcal{G}_N(\partial\Omega_-)^* \langle \tilde{\gamma}_D f_-, \varphi \rangle_{\mathcal{G}_N(\partial\Omega_-)}$$

for all $\varphi \in \mathcal{G}_N(\partial\Omega_\pm)$ such that $\varphi|_{\mathcal{C}_\pm} = 0$.

Proof. The proof consists of three steps. We define the operator

$$Tf := \mathcal{A}f = \begin{pmatrix} -\Delta f_+ \\ \Delta f_- \end{pmatrix},$$

$$\text{dom } T := \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : f_\pm, \Delta f_\pm \in L^2(\Omega_\pm), \tilde{\gamma}_D f_\pm \in \mathcal{G}_\pm^\perp, \tilde{\gamma}_D f_+|_{\mathcal{C}} = \tilde{\gamma}_D f_-|_{\mathcal{C}} \right\},$$

and it will be shown in Step 1 and Step 2 that $T^* = R$. In Step 3 we verify that T is closed, so that

$$R^* = T^{**} = \overline{T} = T.$$

Step 1. We verify that $R \subset T^*$ holds. For this fix some $f = (f_+, f_-)^\top \in \text{dom } R$, and note that $f_\pm = f_{D\pm}$ in the decomposition (2.4)–(2.5). As both T and R are restrictions of the orthogonal sum $S_+^* \oplus S_-^*$ of the maximal operators in (2.3) it follows from Green's identity in Proposition 2.2 that for any $g \in \text{dom } T$ decomposed in the form $g_\pm = g_{D\pm} + g_{0\pm}$ we have

$$\begin{aligned} (Rf, g)_{L^2(\Omega)} - (f, Tg)_{L^2(\Omega)} &= ((S_+^* \oplus S_-^*)f, g)_{L^2(\Omega)} - (f, (S_+^* \oplus S_-^*)g)_{L^2(\Omega)} \\ &= (S_+^* f_+, g_+)_{L^2(\Omega_+)} - (f_+, S_+^* g_+)_{L^2(\Omega_+)} \\ &\quad + (S_-^* f_-, g_-)_{L^2(\Omega_-)} - (f_-, S_-^* g_-)_{L^2(\Omega_-)} \\ &= \mathcal{G}_N(\partial\Omega_+)^* \langle \tilde{\gamma}_D f_+, \gamma_{N+} g_{D+} \rangle_{\mathcal{G}_N(\partial\Omega_+)} - \mathcal{G}_N(\partial\Omega_+) \langle \gamma_{N+} f_{D+}, \tilde{\gamma}_D g_+ \rangle_{\mathcal{G}_N(\partial\Omega_+)^*} \\ &\quad - \mathcal{G}_N(\partial\Omega_-)^* \langle \tilde{\gamma}_D f_-, \gamma_{N-} g_{D-} \rangle_{\mathcal{G}_N(\partial\Omega_-)} + \mathcal{G}_N(\partial\Omega_-) \langle \gamma_{N-} f_{D-}, \tilde{\gamma}_D g_- \rangle_{\mathcal{G}_N(\partial\Omega_-)^*} \\ &= -\mathcal{G}_N(\partial\Omega_+)^* \langle \gamma_{N+} f_+, \tilde{\gamma}_D g_+ \rangle_{\mathcal{G}_N(\partial\Omega_+)^*} + \mathcal{G}_N(\partial\Omega_-) \langle \gamma_{N-} f_-, \tilde{\gamma}_D g_- \rangle_{\mathcal{G}_N(\partial\Omega_-)^*}, \end{aligned}$$

where in the last step we have used that for $f = f_+ \oplus f_- \in \text{dom } R$ one has $f_\pm = f_{D\pm}$, and $f_\pm \in H_0^1(\Omega_\pm)$, so that, $\tilde{\gamma}_D f_\pm = 0$; cf. (2.6). Next we decompose $\gamma_{N\pm} f_\pm$ in the form

$$(3.1) \quad \gamma_{N\pm} f_\pm = (\gamma_{N\pm} f_\pm|_{\mathcal{C}})^\sim + (\gamma_{N\pm} f_\pm|_{\mathcal{C}_\pm})^\sim,$$

where both extensions by 0 on the right hand side belong to the space $\mathcal{G}_N(\partial\Omega_\pm)$ (see Lemma 2.1), and in particular

$$(\gamma_{N\pm} f_{D\pm}|_{\mathcal{C}_\pm})^\sim \in \mathcal{G}_\pm.$$

Since $g \in \text{dom } T$ we have $\tilde{\gamma}_D g_\pm \in \mathcal{G}_\pm^\perp$ and therefore

$$\mathcal{G}_N(\partial\Omega_\pm) \langle (\gamma_{N\pm} f_\pm|_{\mathcal{C}_\pm})^\sim, \tilde{\gamma}_D g_\pm \rangle_{\mathcal{G}_N(\partial\Omega_\pm)^*} = 0.$$

Hence we conclude

$$(3.2) \quad \begin{aligned} (Rf, g)_{L^2(\Omega)} - (f, Tg)_{L^2(\Omega)} &= -\mathcal{G}_N(\partial\Omega_+) \langle (\gamma_{N+} f_+|_{\mathcal{C}})^\sim, \tilde{\gamma}_D g_+ \rangle_{\mathcal{G}_N(\partial\Omega_+)^*} \\ &\quad + \mathcal{G}_N(\partial\Omega_-) \langle (\gamma_{N-} f_-|_{\mathcal{C}})^\sim, \tilde{\gamma}_D g_- \rangle_{\mathcal{G}_N(\partial\Omega_-)^*}. \end{aligned}$$

Since $f \in \text{dom } R$ and $g \in \text{dom } T$ we obtain

$$\gamma_{N+} f_+|_{\mathcal{C}} = \gamma_{N-} f_-|_{\mathcal{C}} \quad \text{and} \quad \tilde{\gamma}_D g_+|_{\mathcal{C}} = \tilde{\gamma}_D g_-|_{\mathcal{C}}.$$

This and (3.2) implies that $(Rf, g)_{L^2(\Omega)} - (f, Tg)_{L^2(\Omega)} = 0$ holds for all $g \in \text{dom } T$. Therefore $f \in \text{dom } T^*$ and $T^* f = Rf$. We have shown $R \subset T^*$.

Step 2. We now verify the opposite inclusion $T^* \subset R$. For this observe first that the orthogonal sum of the Dirichlet operator $A_{D+} \oplus A_{D-}$ is a selfadjoint restriction of T , and hence we have

$$(3.3) \quad T^* \subset A_{D+} \oplus A_{D-} \subset T.$$

Let $f = (f_+, f_-)^\top \in \text{dom } T^*$. Then $f_\pm \in H^2(\Omega_\pm) \cap H_0^1(\Omega_\pm)$ and $f_\pm = f_{D\pm}$ in the decomposition (2.4)–(2.5). It remains to show that the boundary condition

$$(3.4) \quad \gamma_{N+} f_+|_{\mathcal{C}} = \gamma_{N-} f_-|_{\mathcal{C}}$$

is satisfied. For this note that by (2.6) we also have $\tilde{\gamma}_D f_\pm = 0$. For $g \in \text{dom } T$ we obtain in the same way as in Step 1 of the proof that

$$\begin{aligned} 0 &= (T^* f, g)_{L^2(\Omega)} - (f, Tg)_{L^2(\Omega)} \\ &= -\mathcal{G}_N(\partial\Omega_+) \langle \gamma_{N+} f_+, \tilde{\gamma}_D g_+ \rangle_{\mathcal{G}_N(\partial\Omega_+)^*} + \mathcal{G}_N(\partial\Omega_-) \langle \gamma_{N-} f_-, \tilde{\gamma}_D g_- \rangle_{\mathcal{G}_N(\partial\Omega_-)^*}. \end{aligned}$$

Next we decompose $\gamma_{N\pm} f_{\pm}$ as in (3.1) and use that $\tilde{\gamma}_D g_{\pm} \in \mathcal{G}_{\pm}^{\perp}$. As in Step 1 this leads to

$$0 = \mathcal{G}_N(\partial\Omega_+)^* \langle (\gamma_{N+} f_+ |_{\mathcal{C}})^{\sim}, \tilde{\gamma}_D g_+ \rangle_{\mathcal{G}_N(\partial\Omega_+)^*} - \mathcal{G}_N(\partial\Omega_-)^* \langle (\gamma_{N-} f_- |_{\mathcal{C}})^{\sim}, \tilde{\gamma}_D g_- \rangle_{\mathcal{G}_N(\partial\Omega_-)^*}$$

for all $g = (g_+, g_-)^{\top} \in \text{dom } T$. Furthermore, since $\tilde{\gamma}_D g_+ |_{\mathcal{C}} = \tilde{\gamma}_D g_- |_{\mathcal{C}}$ we find

$$0 = \mathcal{G}_N(\partial\Omega_+)^* \langle (\gamma_{N+} f_+ |_{\mathcal{C}})^{\sim} - (\gamma_{N-} f_- |_{\mathcal{C}})^{\sim}, \tilde{\gamma}_D g_+ \rangle_{\mathcal{G}_N(\partial\Omega_+)^*}.$$

This relation holds true for all $g = (g_+, g_-)^{\top} \in \text{dom } T$, and hence for all elements $\psi = \tilde{\gamma}_D g_+ \in \mathcal{G}_+^{\perp}$. Now it follows from (2.7) and the observation below (2.7) that the function

$$(\gamma_{N+} f_+ |_{\mathcal{C}})^{\sim} - (\gamma_{N-} f_- |_{\mathcal{C}})^{\sim}$$

vanishes on \mathcal{C} . Thus the boundary condition (3.4) is satisfied. We have shown $f \in \text{dom } T$ and hence $R^* \subset T$.

Step 3. We show that T is closed. Let $(f_n) \subset \text{dom } T$ such that $f_n \rightarrow f$ and $T f_n \rightarrow h$ for some $f = (f_+, f_-)^{\top}$, $h = (h_+, h_-)^{\top} \in L^2(\Omega)$. Since $T \subset S_+^* \oplus S_-^*$ and $S_+^* \oplus S_-^*$ is closed it follows that

$$f_{\pm} \in \text{dom } S_{\pm}^* \quad \text{and} \quad S_{\pm}^* f_{\pm} = h_{\pm}.$$

Thus it remains to show that the boundary conditions

$$\tilde{\gamma}_D f_{\pm} \in \mathcal{G}_{\pm}^{\perp} \quad \text{and} \quad \tilde{\gamma}_D f_+ |_{\mathcal{C}} = \tilde{\gamma}_D f_- |_{\mathcal{C}}$$

hold. But this follows immediately since $f_{n\pm} \rightarrow f_{\pm}$ in the graph norm of S_{\pm}^* and $\tilde{\gamma}_D$ is continuous with respect to the graph norm, so that, $\tilde{\gamma}_D f_{n\pm} \rightarrow \tilde{\gamma}_D f_{\pm}$ in $(\mathcal{G}_N(\partial\Omega_{\pm}))^*$. \square

The following lemma states that the Neumann traces of the functions from $\ker R^*$ coincide on \mathcal{C} . This property is essentially a consequence of the symmetry of the domain Ω and the function $\text{sgn}(\cdot)$ with respect to the interface \mathcal{C} . For completeness we mention that the functions

$$(3.5) \quad f_{0,k}(x, y) = \begin{cases} \sinh(k\pi(1-x)) \sin(k\pi y), & (x, y) \in \Omega_+, \\ \sinh(k\pi(1+x)) \sin(k\pi y), & (x, y) \in \Omega_-, \end{cases} \quad k \in \mathbb{N} = \{1, 2, \dots\},$$

span a dense set in $\ker R^*$; cf. Proposition 5.1 (iv).

Lemma 3.2. *Let R and R^* be as in Proposition 3.1. Then the following hold.*

- (i) *The space $\ker R^*$ is infinite dimensional and the functions $f_0 \in \ker R^*$ satisfy*

$$(3.6) \quad \tilde{\gamma}_{N+} f_{0+} |_{\mathcal{C}} = \tilde{\gamma}_{N-} f_{0-} |_{\mathcal{C}},$$

that is,

$$\mathcal{G}_D(\partial\Omega_+)^* \langle \tilde{\gamma}_{N+} f_{0+}, \varphi \rangle_{\mathcal{G}_D(\partial\Omega_+)} = \mathcal{G}_D(\partial\Omega_-)^* \langle \tilde{\gamma}_{N-} f_{0-}, \varphi \rangle_{\mathcal{G}_D(\partial\Omega_-)}$$

holds for all $\varphi \in \mathcal{G}_D(\partial\Omega_{\pm})$ such that $\varphi|_{\mathcal{C}_{\pm}} = 0$;

- (ii) *R is invertible and has closed range.*

Proof. (i) As $A_{D+} \oplus A_{D-} \subset R^*$ and $0 \notin \sigma(A_{D\pm})$ we have the direct sum decomposition

$$\text{dom } R^* = \text{dom } (A_{D+} \oplus A_{D-}) \dot{+} \ker R^*.$$

Together with (2.6) this yields that the mapping

$$(3.7) \quad \tilde{\Gamma}_D : \ker R^* \rightarrow \mathcal{G}_N(\partial\Omega_+) \times \mathcal{G}_N(\partial\Omega_-), \quad f_0 = \begin{pmatrix} f_{0+} \\ f_{0-} \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\gamma}_D f_{0+} \\ \tilde{\gamma}_D f_{0-} \end{pmatrix},$$

is invertible. Suppose now that $f_0 = (f_{0+}, f_{0-})^\top \in \ker R^*$ and assume, in addition, that $f_{0\pm} \in H^2(\Omega_\pm)$. Then $\Delta f_{0\pm} = 0$ and the boundary conditions have the explicit form

$$(3.8) \quad \gamma_D f_{0\pm}|_{\mathcal{C}_\pm} = 0 \quad \text{and} \quad \gamma_D f_{0+}|_{\mathcal{C}} = \gamma_D f_{0-}|_{\mathcal{C}};$$

here γ_D is the Dirichlet trace operator defined on $H^2(\Omega_\pm)$. It follows that the function

$$h(x, y) := f_{0+}(-x, y), \quad x \in (-1, 0), \quad y \in (0, 1),$$

belongs to $L^2(\Omega_-)$ and satisfies $\Delta h = 0$ and $\gamma_D h|_{\mathcal{C}} = \gamma_D f_{0+}|_{\mathcal{C}}$ and $\gamma_D h|_{\mathcal{C}_-} = 0$. Hence $(f_{0+}, h)^\top \in \ker R^*$ but as the map $\tilde{\Gamma}_D$ in (3.7) is invertible we conclude $f_{0-} = h$. In particular, if γ_{N_\pm} denotes the Neumann trace operator on $H^2(\Omega_\pm)$ we obtain

$$\gamma_{N_-} f_{0-}|_{\mathcal{C}} = \gamma_{N_-} h|_{\mathcal{C}} = \gamma_{N_+} f_{0+}|_{\mathcal{C}}.$$

As $\tilde{\gamma}_{N_\pm}$ are extensions of γ_{N_\pm} this yields

$$\mathcal{G}_D(\partial\Omega_+)^* \langle \tilde{\gamma}_{N_+} f_{0+}, \varphi \rangle_{\mathcal{G}_D(\partial\Omega_+)} = \mathcal{G}_D(\partial\Omega_-)^* \langle \tilde{\gamma}_{N_-} f_{0-}, \varphi \rangle_{\mathcal{G}_D(\partial\Omega_-)}$$

for all $\varphi \in \mathcal{G}_D(\partial\Omega_\pm)$ such that $\varphi|_{\mathcal{C}_\pm} = 0$. We have shown that any function $f_0 \in \ker R^*$ with the additional property $f_{0\pm} \in H^2(\Omega_\pm)$ satisfies the condition (3.6). The general case follows from $R^* \subset S_+^* \oplus S_-^*$, the fact that $\ker S_\pm^* \cap H^2(\Omega_\pm)$ is dense in $\ker S_\pm^*$ and the continuity of the extended Neumann trace maps $\tilde{\gamma}_{N_\pm}$.

(ii) Since $R \subset A_{D+} \oplus A_{D-} \subset R^*$ and $0 \notin \sigma(A_{D_\pm})$ it follows that $\ker R = \{0\}$. In order to see that $\text{ran } R$ is closed assume that $Rf_n \rightarrow g$, $n \rightarrow \infty$, for some $g \in L^2(\Omega)$. It is clear that also $(A_{D+} \oplus A_{D-})f_n \rightarrow g$, $n \rightarrow \infty$, and from $0 \notin \sigma(A_{D_\pm})$ we conclude

$$f_n \rightarrow f := (A_{D+}^{-1} \oplus A_{D-}^{-1})g, \quad n \rightarrow \infty.$$

Since R is closed we find $f \in \text{dom } R$ and $Rf = g$. \square

4. THE SELFADJOINT OPERATOR A AND ITS QUALITATIVE SPECTRAL PROPERTIES

In this section we present the main result of this note. The operator A (informally written in (1.2)) is now defined rigorously with explicit boundary conditions as a restriction of the maximal operator $S_+^* \oplus S_-^*$. It is shown that A is selfadjoint in $L^2(\Omega)$ and it turns out that A can be viewed as a generalized Krein-von Neumann extension of the non-semibounded symmetric operator R (see also Proposition 4.2 below).

Theorem 4.1. *The operator*

$$(4.1) \quad \begin{aligned} Af = \mathcal{A}f &= \begin{pmatrix} -\Delta f_+ \\ \Delta f_- \end{pmatrix}, \\ \text{dom } A &= \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : \begin{aligned} &f_\pm, \Delta f_\pm \in L^2(\Omega_\pm), \tilde{\gamma}_D f_\pm \in \mathcal{G}_\pm^\perp, \\ &\tilde{\gamma}_D f_+|_{\mathcal{C}} = \tilde{\gamma}_D f_-|_{\mathcal{C}}, \tilde{\gamma}_{N_+} f_+|_{\mathcal{C}} = \tilde{\gamma}_{N_-} f_-|_{\mathcal{C}} \end{aligned} \right\}, \end{aligned}$$

is selfadjoint in $L^2(\Omega)$ and coincides with the operator $R^* \upharpoonright \text{dom } R \dot{+} \ker R^*$. The boundary conditions $\tilde{\gamma}_D f_+|_e = \tilde{\gamma}_D f_-|_e$ and $\tilde{\gamma}_{N_+} f_+|_e = \tilde{\gamma}_{N_-} f_-|_e$ are understood as

$$\mathcal{G}_N(\partial\Omega_+)^* \langle \tilde{\gamma}_D f_+, \varphi \rangle_{\mathcal{G}_N(\partial\Omega_+)} = \mathcal{G}_N(\partial\Omega_-)^* \langle \tilde{\gamma}_D f_-, \varphi \rangle_{\mathcal{G}_N(\partial\Omega_-)}$$

for all $\varphi \in \mathcal{G}_N(\partial\Omega_\pm)$ such that $\varphi|_{e_\pm} = 0$, and

$$\mathcal{G}_D(\partial\Omega_+)^* \langle \tilde{\gamma}_N f_+, \psi \rangle_{\mathcal{G}_D(\partial\Omega_+)} = \mathcal{G}_D(\partial\Omega_-)^* \langle \tilde{\gamma}_N f_-, \psi \rangle_{\mathcal{G}_D(\partial\Omega_-)}$$

for all $\psi \in \mathcal{G}_D(\partial\Omega_\pm)$ such that $\psi|_{e_\pm} = 0$, respectively.

Proof. We first show that $A \subset A^*$ holds. Since $A \subset R^* \subset S_+^* \oplus S_-^*$ we have for $f, g \in \text{dom } A$ decomposed in the usual form $f_\pm = f_{D\pm} + f_{0\pm}$, $g_\pm = g_{D\pm} + g_{0\pm}$ (see (2.4)–(2.5))

$$\begin{aligned} (Af, g)_{L^2(\Omega)} - (f, Ag)_{L^2(\Omega)} &= ((S_+^* \oplus S_-^*)f, g)_{L^2(\Omega)} - (f, (S_+^* \oplus S_-^*)g)_{L^2(\Omega)} \\ &= \mathcal{G}_N(\partial\Omega_+)^* \langle \tilde{\gamma}_D f_+, \gamma_{N_+} g_{D+} \rangle_{\mathcal{G}_N(\partial\Omega_+)} - \mathcal{G}_N(\partial\Omega_+)^* \langle \gamma_{N_+} f_{D+}, \tilde{\gamma}_D g_+ \rangle_{\mathcal{G}_N(\partial\Omega_+)^*} \\ &\quad - \mathcal{G}_N(\partial\Omega_-)^* \langle \tilde{\gamma}_D f_-, \gamma_{N_-} g_{D-} \rangle_{\mathcal{G}_N(\partial\Omega_-)} + \mathcal{G}_N(\partial\Omega_-)^* \langle \gamma_{N_-} f_{D-}, \tilde{\gamma}_D g_- \rangle_{\mathcal{G}_N(\partial\Omega_-)^*}; \end{aligned}$$

cf. Proposition 2.2 and Step 1 in the proof of Proposition 3.1. Taking into account $\tilde{\gamma}_D f_\pm, \tilde{\gamma}_D g_\pm \in \mathcal{G}_\pm^\perp$ and decomposing $\gamma_{N_\pm} f_{D\pm}$ and $\gamma_{N_\pm} g_{D\pm}$ in the form

$$\begin{aligned} \gamma_{N_\pm} f_{D\pm} &= (\gamma_{N_\pm} f_{D\pm}|_e)^\sim + (\gamma_{N_\pm} f_{D\pm}|_{e_\pm})^\sim, \\ \gamma_{N_\pm} g_{D\pm} &= (\gamma_{N_\pm} g_{D\pm}|_e)^\sim + (\gamma_{N_\pm} g_{D\pm}|_{e_\pm})^\sim, \end{aligned}$$

where the extensions by 0 on the right hand side belong to the spaces $\mathcal{G}_N(\partial\Omega_\pm)$ by Lemma 2.1, we find that

$$\begin{aligned} (Af, g)_{L^2(\Omega)} - (f, Ag)_{L^2(\Omega)} &= \mathcal{G}_N(\partial\Omega_+)^* \langle \tilde{\gamma}_D f_+, (\gamma_{N_+} g_{D+}|_e)^\sim \rangle_{\mathcal{G}_N(\partial\Omega_+)} \\ &\quad - \mathcal{G}_N(\partial\Omega_+)^* \langle (\gamma_{N_+} f_{D+}|_e)^\sim, \tilde{\gamma}_D g_+ \rangle_{\mathcal{G}_N(\partial\Omega_+)^*} \\ &\quad - \mathcal{G}_N(\partial\Omega_-)^* \langle \tilde{\gamma}_D f_-, (\gamma_{N_-} g_{D-}|_e)^\sim \rangle_{\mathcal{G}_N(\partial\Omega_-)} \\ &\quad + \mathcal{G}_N(\partial\Omega_-)^* \langle (\gamma_{N_-} f_{D-}|_e)^\sim, \tilde{\gamma}_D g_- \rangle_{\mathcal{G}_N(\partial\Omega_-)^*}. \end{aligned}$$

As $f, g \in \text{dom } A$ we also have

$$\tilde{\gamma}_D f_+|_e = \tilde{\gamma}_D f_-|_e \quad \text{and} \quad \tilde{\gamma}_D g_+|_e = \tilde{\gamma}_D g_-|_e$$

and hence the terms on the right hand side simplify to

$$\begin{aligned} (4.2) \quad &\mathcal{G}_N(\partial\Omega_+)^* \langle \tilde{\gamma}_D f_+, (\gamma_{N_+} g_{D+}|_e)^\sim - (\gamma_{N_-} g_{D-}|_e)^\sim \rangle_{\mathcal{G}_N(\partial\Omega_+)} \\ &- \mathcal{G}_N(\partial\Omega_+)^* \langle (\gamma_{N_+} f_{D+}|_e)^\sim - (\gamma_{N_-} f_{D-}|_e)^\sim, \tilde{\gamma}_D g_+ \rangle_{\mathcal{G}_N(\partial\Omega_+)^*}. \end{aligned}$$

According to Lemma 3.2 (ii) the functions $f_{0\pm}, g_{0\pm} \in \ker R^*$ satisfy

$$\tilde{\gamma}_{N_+} f_{0+}|_e = \tilde{\gamma}_{N_-} f_{0-}|_e \quad \text{and} \quad \tilde{\gamma}_{N_+} g_{0+}|_e = \tilde{\gamma}_{N_-} g_{0-}|_e.$$

Thus we have

$$\begin{aligned} 0 &= \tilde{\gamma}_{N_+} f_+|_e - \tilde{\gamma}_{N_-} f_-|_e = \tilde{\gamma}_{N_+} (f_{D+} + f_{0+})|_e - \tilde{\gamma}_{N_-} (f_{D-} + f_{0-})|_e \\ &= \gamma_{N_+} f_{D+}|_e - \gamma_{N_-} f_{D-}|_e \end{aligned}$$

and

$$\begin{aligned} 0 &= \tilde{\gamma}_{N_+} g_+|_e - \tilde{\gamma}_{N_-} g_-|_e = \tilde{\gamma}_{N_+} (g_{D+} + g_{0+})|_e - \tilde{\gamma}_{N_-} (g_{D-} + g_{0-})|_e \\ &= \gamma_{N_+} g_{D+}|_e - \gamma_{N_-} g_{D-}|_e, \end{aligned}$$

and hence the corresponding entries in (4.2) vanish, that is,

$$(Af, g)_{L^2(\Omega)} - (f, Ag)_{L^2(\Omega)} = 0, \quad f, g \in \text{dom } A.$$

We have shown that $A \subset A^*$ holds.

Next we verify that the operator

$$R_0 := R^* \upharpoonright \text{dom } R \dot{+} \ker R^*$$

is contained in A . In fact, the inclusion $\text{dom } R \subset \text{dom } A$ is obvious and hence it remains to show that $\ker R^* \subset \text{dom } A$. It is clear from the definition of $\text{dom } R^*$ that any function in $f_0 = (f_{0+}, f_{0-}) \in \ker R^*$ satisfies the boundary conditions for functions in $\text{dom } A$, with the exception of the condition $\tilde{\gamma}_{N+} f_{0+}|_c = \tilde{\gamma}_{N-} f_{0-}|_c$. But this last condition holds by Lemma 3.2 (i). Therefore $R_0 \subset A$. We claim that R_0 is selfadjoint. First of all R_0 is symmetric since for $f = f_R + f_0 \in \text{dom } R \dot{+} \ker R^*$ one has

$$(R_0 f, f)_{L^2(\Omega)} = (R_0(f_R + f_0), f_R + f_0)_{L^2(\Omega)} = (R f_R, f_R)_{L^2(\Omega)},$$

and R is symmetric. Moreover, by Lemma 3.2 (ii) 0 is a point of regular type of R , that is,

$$\ker R = \{0\} \quad \text{and} \quad \text{ran } R \quad \text{is closed.}$$

This leads to the direct sum decomposition

$$\text{ran } (R_0 - \mu) = \text{ran } (R - \mu) \dot{+} \ker R^* = L^2(\Omega), \quad \mu \in \mathbb{C} \setminus \mathbb{R},$$

from which we then conclude that R_0 is a selfadjoint operator in $L^2(\Omega)$. Summing up we have shown that A is a symmetric operator which contains the selfadjoint operator R_0 , so that $A = R_0$ is selfadjoint. \square

Finally we state a result on the spectral properties of the operator A . Our proof is a variant of [1, Lemma 2.3], see also [22].

Proposition 4.2. *Let A be the selfadjoint operator from Theorem 4.1. Then 0 is an isolated eigenvalue of infinite multiplicity and the corresponding eigenspace is given by $\ker R^*$. The spectrum in $\mathbb{R} \setminus \{0\}$ is discrete (i.e. composed of isolated eigenvalues of finite multiplicities) and accumulates to $+\infty$ and $-\infty$.*

Proof. It is clear that the eigenspace $\ker A = \ker R^*$ is an infinite dimensional closed subspace of $L^2(\Omega)$. Moreover,

$$(4.3) \quad \mathcal{H} := \text{ran } A = (\ker A)^\perp = (\ker R^*)^\perp = \text{ran } R$$

is closed according to Lemma 3.2 (ii). In the following we denote the orthogonal projection onto the subspace \mathcal{H} by P and the embedding of \mathcal{H} into $L^2(\Omega)$ is denoted by ι . For the restriction of A to \mathcal{H} we write A' . Note that A' is a bijective selfadjoint operator in the Hilbert space \mathcal{H} , so that $0 \notin \sigma(A')$. With respect to the decomposition $L^2(\Omega) = \mathcal{H} \oplus \mathcal{H}^\perp$ we have $A = A' \oplus 0$ and hence

$$(4.4) \quad Af = \iota A' P f, \quad f \in \text{dom } A.$$

It will also be used below that the orthogonal sum $A_D = A_{D+} \oplus A_{D-}$ of the Dirichlet operators $A_{D\pm}$ is a selfadjoint operator in $L^2(\Omega)$ and that $0 \notin \sigma(A_D)$.

Let now $f = f_R + f_0 \in \text{dom } A$, where $f_R \in \text{dom } R$ and $f_0 \in \ker A$. As $R \subset A_D$ and $R \subset A$ we have

$$f = f_R + f_0 = A_D^{-1} R f_R + f_0 = A_D^{-1} A f_R + f_0 = A_D^{-1} A f + f_0$$

and hence

$$Pf = P(A_D^{-1}Af + f_0) = PA_D^{-1}Af = PA_D^{-1}\iota A'Pf,$$

where we have used (4.4) in the last equality. This leads to

$$A'^{-1}(A'Pf) = Pf = PA_D^{-1}\iota(A'Pf)$$

and as $0 \notin \sigma(A')$ we conclude

$$A'^{-1} = PA_D^{-1}\iota.$$

Since A_D^{-1} is a compact operator in $L^2(\Omega)$ it follows that A'^{-1} is a compact operator in \mathcal{H} . Moreover, for $g \in \mathcal{H}$ we have

$$(4.5) \quad (A'^{-1}g, g)_{\mathcal{H}} = (PA_D^{-1}\iota g, g)_{\mathcal{H}} = (A_D^{-1}\iota g, \iota g)_{L^2(\Omega)}.$$

Since $S_+ \oplus S_- \subset R$ we conclude for all $f_{\pm} \in \text{dom } S_{\pm} = H_0^2(\Omega_{\pm})$

$$(S_+f_+, 0)^{\top} \in \text{ran } R = \mathcal{H} \quad \text{and} \quad (0, S_-f_-)^{\top} \in \text{ran } R = \mathcal{H}.$$

It follows that the spaces $\mathcal{H} \cap L^2(\Omega_{\pm})$ are both infinite dimensional. It is clear that the form on the right hand side of (4.5) is positive (negative) for functions in $\mathcal{H} \cap L^2(\Omega_+)$ (resp. $\mathcal{H} \cap L^2(\Omega_-)$). This implies that the positive and negative spectra of A'^{-1} are both infinite. Now it follows from the compactness that the spectrum of A' (and hence of A) in $\mathbb{R} \setminus \{0\}$ is discrete and accumulates to $+\infty$ and $-\infty$. \square

5. QUANTITATIVE SPECTRAL PROPERTIES OF THE SELFADJOINT OPERATOR A

According to Proposition 4.2 the spectrum of the selfadjoint operator A consists of eigenvalues which accumulate to $+\infty$ and $-\infty$. The eigenvalue 0 is of infinite multiplicity, the multiplicities of the nonzero eigenvalues are finite. In the next proposition we identify the eigenvalues of A with the roots of an elementary algebraic equation and we specify the eigenfunctions of A .

Proposition 5.1. *Let A be the selfadjoint operator from Theorem 4.1. Then the following hold.*

- (i) *The spectrum of A is symmetric with respect to 0.*
- (ii) *We have*

$$\sigma(A) = \bigcup_{n=1}^{\infty} \bigcup_{m=-\infty}^{\infty} \{\lambda_{n,m}\},$$

where $\{\lambda_{n,m}\}_{m \in \mathbb{Z}}$ for each fixed $n \in \mathbb{N}$ is an increasing sequence of simple roots of the algebraic equation

$$(5.1) \quad \frac{\tanh \sqrt{\lambda + (n\pi)^2}}{\sqrt{\lambda + (n\pi)^2}} = \frac{\tan \sqrt{\lambda - (n\pi)^2}}{\sqrt{\lambda - (n\pi)^2}}$$

for $\lambda \neq \pm(n\pi)^2$. We arrange the sequence in such a way that $\lambda_{n,0} = 0$ (zero is a solution of (5.1) for any $n \in \mathbb{N}$).

- (iii) *Given any $n \in \mathbb{N}$, (5.1) has no root in $(-(n\pi)^2, 0) \cup (0, (n\pi)^2)$. In particular, $[-\pi^2, 0) \cup (0, \pi^2] \notin \sigma(A)$.*
- (iv) *The eigenfunction of A corresponding to $\lambda_{n,m}$ is given by $f_{n,m}(x, y) = \psi_{n,m}(x)\chi_n(y)$, where $\chi_n(y) = \sqrt{2}\sin(n\pi y)$ and*

$$(5.2) \quad \psi_{n,m}(x) = \begin{cases} N_{n,m} \sinh \sqrt{\lambda_{n,m} + (n\pi)^2} \sin \left(\sqrt{\lambda_{n,m} - (n\pi)^2} (1-x) \right), & x > 0, \\ N_{n,m} \sin \sqrt{\lambda_{n,m} - (n\pi)^2} \sinh \left(\sqrt{\lambda_{n,m} + (n\pi)^2} (1+x) \right), & x < 0, \end{cases}$$

with any $N_{n,m} \in \mathbb{C} \setminus \{0\}$. With the normalization constants $N_{n,m}$ satisfying

$$|N_{n,m}|^{-2} = \sinh^2 \sqrt{\lambda_{n,m} + (n\pi)^2} \left[\frac{1}{2} - \frac{\sin \left(2\sqrt{\lambda_{n,m} - (n\pi)^2} \right)}{4\sqrt{\lambda_{n,m} - (n\pi)^2}} \right] \\ + \sin^2 \sqrt{\lambda_{n,m} - (n\pi)^2} \left[-\frac{1}{2} + \frac{\sinh \left(2\sqrt{\lambda_{n,m} + (n\pi)^2} \right)}{4\sqrt{\lambda_{n,m} + (n\pi)^2}} \right],$$

the functions $f_{n,m}$ ($n \in \mathbb{N}$, $m \in \mathbb{Z}$) form a complete orthonormal set in $L^2(\Omega)$.

Proof. The eigenvalues λ and the corresponding eigenfunctions f of A can be obtained as nontrivial solutions of the differential equations $\mp \Delta f_{\pm} = \lambda f_{\pm}$ in Ω_{\pm} , subject to the boundary and interface conditions determined in (4.1). From this boundary transmission problem, it is immediately seen that if λ is an eigenvalue of A (with eigenfunction $f(x, y)$), then also $-\lambda$ is an eigenvalue of A (with eigenfunction $f(-x, y)$). This establishes (i).

The other properties (ii)–(iv) are obtained by a separation of variables. Decomposing any eigenfunction $f \in L^2(\Omega)$ of A into the transverse orthonormal Dirichlet basis $\{\chi_n\}_{n=1}^{\infty}$, i.e.,

$$f(x, y) = \sum_{n=1}^{\infty} \psi_n(x) \chi_n(y), \quad \chi_n(y) = \sqrt{2} \sin(n\pi y),$$

we easily obtain from the boundary transmission problem in Ω that the function $\psi_n = (\psi_{n+}, \psi_{n-})^{\top} \in L^2((0, 1)) \times L^2((-1, 0))$ for each fixed $n \in \mathbb{N}$ is a nontrivial solution of the following problem

$$(5.3) \quad \begin{aligned} -\psi_{n+}'' &= (\lambda - (n\pi)^2) \psi_{n+} & \text{in } (0, 1), \\ \psi_{n-}'' &= (\lambda + (n\pi)^2) \psi_{n-} & \text{in } (-1, 0), \end{aligned}$$

subject to the boundary and interface conditions

$$(5.4) \quad \psi_{n+}(1) = \psi_{n-}(-1) = 0, \quad \psi_{n+}(0) = \psi_{n-}(0), \quad \text{and} \quad \psi_{n+}'(0) = -\psi_{n-}'(0).$$

Solving the differential equations in (5.3) in terms of exponentials and subjecting the latter to the boundary and interface conditions (5.4), we find that any nontrivial solution ψ_n is of the form (5.2) with the constrain that the eigenvalue λ solves (5.1). There is an infinite number of such solutions because (5.1) always contains an oscillatory tangent function for large values of λ . For each fixed $n \in \mathbb{N}$, we arrange the roots of (5.1) in an increasing sequence $\{\lambda_{n,m}\}_{m \in \mathbb{Z}}$ such that $\lambda_{n,0} = 0$. Notice that $\lambda = \pm(n\pi)^2$ are not admissible solutions of (5.3) for any $n \in \mathbb{N}$. This is in fact consistent with (5.1), because the limit $\lambda \rightarrow \pm(n\pi)^2$ casts (5.1) into $\tanh \sqrt{2(n\pi^2)} = \sqrt{2(n\pi^2)}$ which is never satisfied for nonzero n . We have thus proved (ii), except for the simplicity of the roots of (5.1), which will be established at the end of this proof. As for (iv), it only remains to recall that eigenfunctions of a selfadjoint operator with pure point spectrum form a complete orthonormal set when normalized properly ($N_{n,m}$ is chosen in such a way that all $\psi_{n,m}$ have norm 1 in $L^2((-1, 1))$ and χ_n are already normalized to 1 in $L^2((0, 1))$).

Now we turn to a proof of (iii). Recall that we already know that no eigenvalue can be equal to $\pm(n\pi)^2$, with $n \in \mathbb{N}$. To show that (5.1) has no root in $(0, (n\pi)^2)$,

it is enough to show that the function

$$G(\lambda) = \frac{\sqrt{\lambda + (n\pi)^2}}{\tanh \sqrt{\lambda + (n\pi)^2}} - \frac{\sqrt{(n\pi)^2 - \lambda}}{\tanh \sqrt{(n\pi)^2 - \lambda}}$$

does not vanish in $(0, (n\pi)^2)$. This follows from $G(0) = 0$ and

$$G'(\lambda) = \frac{1}{4} \left[\frac{\sinh \left(2\sqrt{\lambda + (n\pi)^2} \right) - 2\sqrt{\lambda + (n\pi)^2}}{\sqrt{\lambda + (n\pi)^2} \sinh^2 \sqrt{\lambda + (n\pi)^2}} + \frac{\sinh \left(2\sqrt{(n\pi)^2 - \lambda} \right) - 2\sqrt{(n\pi)^2 - \lambda}}{\sqrt{(n\pi)^2 - \lambda} \sinh^2 \sqrt{(n\pi)^2 - \lambda}} \right] > 0,$$

for $\lambda \in (0, (n\pi)^2)$, where the crucial inequality is due to the elementary bound $\sinh(x) > x$ valid for all $x > 0$. Since (5.1) is symmetric with respect to the change $\lambda \mapsto -\lambda$, the claim on the absence of roots extends to the symmetric set $-(n\pi)^2, 0) \cup (0, (n\pi)^2)$.

It remains to prove the simplicity of roots stated in (ii). By symmetry of (5.1), it is again enough to show it for non-negative roots $\lambda_{n,m}$ only. Defining

$$(5.5) \quad F(\lambda) = \frac{\tanh \sqrt{\lambda + (n\pi)^2}}{\sqrt{\lambda + (n\pi)^2}} - \frac{\tanh \sqrt{\lambda - (n\pi)^2}}{\sqrt{\lambda - (n\pi)^2}},$$

we have that $\lambda_{n,m}$ is a root of (5.1) if, and only if, $F(\lambda_{n,m}) = 0$. Using this identity, it is straightforward to cast the derivative of F at $\lambda_{n,m}$ into the form

$$F'(\lambda_{n,m}) = -\frac{\tanh^2 \sqrt{\lambda_{n,m} + (n\pi)^2}}{\lambda_{n,m} + (n\pi)^2} + \frac{(n\pi)^2}{\lambda_{n,m}^2 - (n\pi)^4} \left[\frac{\tanh \sqrt{\lambda_{n,m} + (n\pi)^2}}{\sqrt{\lambda_{n,m} + (n\pi)^2}} - 1 \right].$$

If $\lambda_{n,m} > 0$, then we know by (iii) that necessarily $\lambda_{n,m} > (n\pi)^2$. Using the elementary bound $\tanh(x) < x$ for all $x > 0$, we thus obtain

$$F'(\lambda_{n,m}) < -\frac{\tanh^2 \sqrt{\lambda_{n,m} + (n\pi)^2}}{\lambda_{n,m} + (n\pi)^2} < 0.$$

On the other hand, employing standard algebraic expressions for hyperbolic functions, it is easy to check that the formula for $F'(\lambda_{n,m})$ above reduces for $\lambda_{n,0} = 0$ to

$$F'(0) = \frac{2n\pi - \sinh(2n\pi)}{2(n\pi)^3 \cosh^2(n\pi)} < 0,$$

where the inequality follows by the elementary bound used above in the proof of (iii). Summing up, $F'(\lambda) \neq 0$ whenever $F(\lambda) = 0$, which proves the simplicity of the roots of (5.1) and completes the proof of the proposition. \square

We remark that the simplicity of roots of (5.1) stated in point (ii) of the above proposition does not mean that the eigenvalues of A are simple. In fact, we already know from Proposition 4.2 that 0 is an eigenvalue of infinite multiplicity.

In order to establish the convergence results announced in the introduction in a unified way, we consider now a more general situation of the differential expression

$$(5.6) \quad \mathcal{J}_\delta f = -\operatorname{div}(a_\delta \nabla f), \quad a_\delta(x, y) = \begin{cases} 1, & (x, y) \in \Omega_+, \\ -\frac{1}{1+\delta}, & (x, y) \in \Omega_-, \end{cases}$$

where δ is an arbitrary complex number with $|\delta| < 1$. We also introduce an associated operator

$$(5.7) \quad T_\delta f = \mathcal{T}_\delta f, \quad \text{dom } T_\delta = \{f \in H_0^1(\Omega) : \mathcal{T}_\delta f \in L^2(\Omega)\}.$$

Clearly, by choosing δ appropriately, the eigenvalue problems for the selfadjoint operator A_ε from (1.4) and the (up to a rotation) m -sectorial operator B_η from (1.6) can be cast into the form of the eigenvalue problem for T_δ . The latter reads

$$(5.8) \quad \begin{aligned} -\Delta f_+ &= \lambda f_+ & \text{in } \Omega_+, \\ \Delta f_- &= (1 + \delta)\lambda f_- & \text{in } \Omega_-, \end{aligned}$$

where, in addition, $f = (f_+, f_-)^\top \in \text{dom } T_\delta \subset H_0^1(\Omega)$ satisfies the interface condition

$$(5.9) \quad (1 + \delta)\partial_{\mathbf{n}_+} f_+|_{\mathcal{C}} = \partial_{\mathbf{n}_-} f_-|_{\mathcal{C}}.$$

Proposition 5.2. *Let T_δ be the operator introduced in (5.7). There exists an absolute constant $c > 0$ such that for $|\delta| \leq c$ the following hold.*

(i) *We have*

$$\sigma_p(T_\delta) = \bigcup_{n=1}^{\infty} \bigcup_{m=-\infty}^{\infty} \{\lambda_{n,m}^\delta\},$$

where $\{\lambda_{n,m}^\delta\}_{m \in \mathbb{Z}}$ for each fixed $n \in \mathbb{N}$ is a sequence of roots of the algebraic equation

$$(5.10) \quad (1 + \delta) \frac{\tanh \sqrt{(1 + \delta)\lambda + (n\pi)^2}}{\sqrt{(1 + \delta)\lambda + (n\pi)^2}} = \frac{\tan \sqrt{\lambda - (n\pi)^2}}{\sqrt{\lambda - (n\pi)^2}}$$

for $\lambda \neq (n\pi)^2$ and $\lambda \neq -(n\pi)^2/(1 + \delta)$.

(ii) *The eigenfunction of T_δ corresponding to $\lambda_{n,m}^\delta$ is given by $f_{n,m}^\delta(x, y) = \psi_{n,m}^\delta(x)\chi_n(y)$, where $\chi_n(y) = \sqrt{2}\sin(n\pi y)$ and*

$$(5.11) \quad \psi_{n,m}^\delta(x) = \begin{cases} N_{n,m}^\delta \sinh \sqrt{(1 + \delta)\lambda_{n,m}^\delta + (n\pi)^2} \sin \left(\sqrt{\lambda_{n,m}^\delta - (n\pi)^2} (1 - x) \right), & x > 0, \\ N_{n,m}^\delta \sin \sqrt{\lambda_{n,m}^\delta - (n\pi)^2} \sinh \left(\sqrt{(1 + \delta)\lambda_{n,m}^\delta + (n\pi)^2} (1 + x) \right), & x < 0, \end{cases}$$

with any $N_{n,m}^\delta \in \mathbb{C} \setminus \{0\}$. With the normalization constants $N_{n,m}^\delta$ satisfying

$$\begin{aligned} |N_{n,m}^\delta|^{-2} &= \left| \sinh \left(\sqrt{(1 + \delta)\lambda_{n,m}^\delta + (n\pi)^2} \right) \right|^2 \\ &\times \left[\frac{\sinh \left(2 \operatorname{Im} \sqrt{\lambda_{n,m}^\delta - (n\pi)^2} \right)}{4 \operatorname{Im} \sqrt{\lambda_{n,m}^\delta - (n\pi)^2}} - \frac{\sin \left(2 \operatorname{Re} \sqrt{\lambda_{n,m}^\delta - (n\pi)^2} \right)}{4 \operatorname{Re} \sqrt{\lambda_{n,m}^\delta - (n\pi)^2}} \right] \\ &+ \left| \sin \left(\sqrt{\lambda_{n,m}^\delta - (n\pi)^2} \right) \right|^2 \\ &\times \left[-\frac{\sin \left(2 \operatorname{Im} \sqrt{(1 + \delta)\lambda_{n,m}^\delta + (n\pi)^2} \right)}{4 \operatorname{Im} \sqrt{(1 + \delta)\lambda_{n,m}^\delta + (n\pi)^2}} + \frac{\sinh \left(2 \operatorname{Re} \sqrt{(1 + \delta)\lambda_{n,m}^\delta + (n\pi)^2} \right)}{4 \operatorname{Re} \sqrt{(1 + \delta)\lambda_{n,m}^\delta + (n\pi)^2}} \right], \end{aligned}$$

the functions $f_{n,m}^\delta$ ($n \in \mathbb{N}$, $m \in \mathbb{Z}$) are normalized to 1 in $L^2(\Omega)$.

Proof. The results follow by the separation of variables as in the proof of Proposition 5.1. Contrary to the symmetric situation $\delta = 0$, however, (5.10) can have solutions $\lambda = (n\pi)^2$ and $\lambda = -(n\pi)^2/(1 + \delta)$. Compatibility conditions for the existence of such solutions are

$$(5.12) \quad \frac{\tanh \sqrt{(2 + \delta)(n\pi)^2}}{\sqrt{(2 + \delta)(n\pi)^2}} = \frac{1}{1 + \delta}, \quad \frac{\tanh \sqrt{\frac{2 + \delta}{1 + \delta}}(n\pi)^2}{\sqrt{\frac{2 + \delta}{1 + \delta}}(n\pi)^2} = 1 + \delta,$$

respectively (they can be obtained from (5.10) after the limit $\lambda \rightarrow (n\pi)^2$ and $\lambda \rightarrow -(n\pi)^2/(1 + \delta)$, respectively). We claim that these “exceptional” solutions do not exist for all δ small in the absolute value, uniformly in $n \in \mathbb{N}$. This can be proved straightforwardly by comparing the real parts of the left and right sides of (5.12). More specifically, we have

$$\left| \operatorname{Re} \left(\frac{\tanh z}{z} \right) \right| = \frac{1}{|z|^2} \left| \frac{z_1 \sinh(2z_1) + z_2 \sin(2z_2)}{\cosh(2z_1) + \cos(2z_2)} \right| \leq \frac{1}{|z_1|} \frac{\sinh(2|z_1|) + 1}{\cosh(2|z_1|) - 1}$$

for all $z = z_1 + iz_2 \in \mathbb{C}$ with $z_1 = \operatorname{Re} z \neq 0$, where the right hand side is decreasing as a function of $|z_1|$. Employing the elementary inequality $|\operatorname{Re} \sqrt{\xi}| \geq |\sqrt{\operatorname{Re} \xi}|$ valid for every $\xi \in \mathbb{C}$ with $\operatorname{Re} \xi \geq 0$ and $|\delta| < 1$, we estimate

$$\begin{aligned} \left| \operatorname{Re} \sqrt{(2 + \delta)(n\pi)^2} \right| &\geq \left| \sqrt{(2 + \operatorname{Re} \delta)(n\pi)^2} \right| \geq \pi, \\ \left| \operatorname{Re} \sqrt{\frac{2 + \delta}{1 + \delta}}(n\pi)^2 \right| &\geq \left| \sqrt{\left(1 + \frac{1 + \operatorname{Re} \delta}{1 + 2\operatorname{Re} \delta + |\delta|^2}\right)}(n\pi)^2 \right| \geq \pi. \end{aligned}$$

Consequently, we see that a necessary condition for an equality to hold in (5.12) is

$$0.32 \approx \frac{1}{\pi} \frac{\sinh(2\pi) + 1}{\cosh(2\pi) - 1} \geq \min \left\{ \operatorname{Re} \left(\frac{1}{1 + \delta} \right), \operatorname{Re}(1 + \delta) \right\} \geq \frac{1 - |\delta|}{(1 + |\delta|)^2},$$

which is clearly impossible if $|\delta|$ is small enough (the present estimates yield $c \geq 0.38$). \square

Now we are in a position to establish the convergence of eigenvalues and eigenfunctions of T_δ to eigenvalues and eigenfunctions of A as $\delta \rightarrow 0$. In the next theorem we show, in particular, that the operators A_ε and B_η in the introduction represent an “approximation” of the selfadjoint operator A , at least on the spectral level. However, the resolvents of A_ε and B_η are compact for all $\varepsilon \neq 1$ and $\eta > 0$, while the resolvent of A is not compact (zero is an eigenvalue of infinite multiplicity).

Theorem 5.3. *For $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, let $\lambda_{n,m}$ and $\psi_{n,m}$ be respectively the eigenvalues and eigenfunctions of A specified in Proposition 5.1 and let $\lambda_{n,m}^\delta$ and $\psi_{n,m}^\delta$ be respectively the eigenvalues and eigenfunctions of T_δ specified in Proposition 5.2. For any $n \in \mathbb{N}$, the sequence $\{\lambda_{n,m}^\delta\}_{m \in \mathbb{Z}}$ can be arranged in such a way that*

$$\lim_{\delta \rightarrow 0} |\lambda_{n,m}^\delta - \lambda_{n,m}| = 0 \quad \text{and} \quad \lim_{\delta \rightarrow 0} \|\psi_{n,m}^\delta - \psi_{n,m}\|_{L^\infty(\Omega)} = 0.$$

Proof. The convergence of eigenvalues follows by the implicit function theorem applied to

$$H(\lambda, \delta) = (1 + \delta) \frac{\tanh \sqrt{(1 + \delta)\lambda + (n\pi)^2}}{\sqrt{(1 + \delta)\lambda + (n\pi)^2}} - \frac{\tan \sqrt{\lambda - (n\pi)^2}}{\sqrt{\lambda - (n\pi)^2}}.$$

Clearly, $H(\lambda, 0) = F(\lambda)$, where F is introduced in (5.5) based on (5.1). Hence, $H(\lambda_{n,m}, 0) = 0$. We only need to check that the derivative $\partial_1 H(\lambda_{n,m}, 0)$ does not vanish. However, $\partial_1 H(\lambda_{n,m}, 0) = F'(\lambda_{n,m}) \neq 0$, due to the proof of simplicity of the roots of (5.1) established in the proof of Proposition 5.1. The convergence of eigenfunctions is then clear from the expressions (5.2) and (5.11). \square

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